A note on Class A Bézier curves

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Abstract

The *Class A* Bézier curves presented in Farin (2006) were constructed by so-called *Class A* matrix, which are special matrices satisfying two appropriate conditions. The speciality of the Class A matrix causes the Class A Bézier to possess two properties, which are sufficient conditions for the proof of the curvature and torsion monotonicity. In this paper, we discover that, in Farin (2006), the conditions Class A matrix satisfied cannot guarantee one of the two properties of the Class A Bézier curves, then the proof of the curvature and torsion monotonicity becomes incomplete. Furthermore, we modified the conditions for the Class A matrices to complete the proof.

Keywords: Class A; Bézier; Monotonicity; Curvature; Torsion

In Farin (2006), the author proposes the construction of Class A Bézier curves, which have monotone curvature and torsion. A Bézier curve

$$x(t) = \sum_{i=0}^{n} b_i B_i^n(t) \tag{1}$$

is defined as *Class A* Bézier curve if the derivative can be written as

$$x'(t) = n \sum_{i=0}^{n-1} M^i v B_i^{n-1}(t), \tag{2}$$

where b_i is the control point, $v = b_1 - b_0 = (x_1, x_2, x_3)$ is the first leg of control polygon (hereinafter, we remove the last element in the planar case) and M is a so-called *Class A matrix*, i.e., its singular values $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ satisfy conditions:

$$\left\|(1-t)v^* + tMv\right\| \geqslant \|v^*\| \quad \text{for } t \in [0,1],\ \|v^*\| = 1, \tag{3}$$

and

$$\sigma_3^2 \geqslant \sigma_1. \tag{4}$$

The Class A Bézier curves hold the following two properties, which are actually sufficient conditions for the proof of the curvature and torsion monotonicity:

Property 1. *Class A curves are invariant under subdivision, i.e., if a Class A curve is subdivided, both resulting curves are Class A curves.*

Property 2. *For a Class A Bézier curve, the curvature (torsion) at the beginning point is always not smaller than the one at the end point.*

Following the first property, any two points $x(t_1)$ and $x(t_2)$ ($t_1 > t_2$) of the Class A curve (1) can respectively be converted to the beginning and end points of a new Class A curve by subdividing the original one twice. Then based on the second property, we can assert that the curvature (torsion) of Class A curve at a smaller parameter value is always larger than the one at a larger parameter value. In other words, the curvature and torsion of the Class A curve are decreasing with respect to the parameter t. It is illustrated in Farin (2006) that the segments generated by subdivision have their derivatives being the form (2), with corresponding matrix becoming

$$T = (1-t)I + tM \quad \text{and} \quad MT^{-1}. \tag{5}$$

Note that the first property is equivalent to that the Class A matrix is invariant under subdivision, namely, the matrices T and MT^{-1} should also satisfy conditions (3) and (4). However, in Farin (2006), only the condition (3) is verified to be satisfied by both T and MT^{-1}, while the condition (4) is not considered. Actually, the matrices in (5) may violate condition (4) after subdivision. We give a counter example in the following.

Counter example. Supposing diagonal matrix $M = (1.102, 1.101, 1.05)$, the matrix T at $t = 0.5$ becomes to $T = (1.051, 1.0505, 1.025)$, which does not satisfy condition (4).

To fix this problem, we propose new conditions for a matrix to be a Class A matrix. Then we verify that the curve induced by the Class A matrix satisfying new conditions will satisfy both properties mentioned above. Then it is sufficiently proved that curves have decreased varying curvature and torsion. For simplicity, we only consider the conditions for symmetric matrices here. We note that the symmetric matrix can be decomposed as $M = SDS^{-1}$, where S and D are orthogonal and diagonal, respectively. Hence, Eq. (2) can be rewritten as

$$x'(t) = nS\sum_{i=0}^{n-1} D^i v^* B_i^{n-1}(t),$$

where $v^* = S^{-1}v$. Since Bézier curves are geometric invariable under translation and rotation, the study of the Class A Bézier curve induced by a diagonal matrix will be sufficient.

First, we present some notation as below, some follow from Farin (2006).

- $\alpha_i = 1 - t + t\sigma_i$, $\beta_i = \frac{\sigma_i}{\alpha_i}$, $i = 1, 2, 3$, the entries of T and MT^{-1};
- $\|v_i\|$: the modulus of $v_i = M^i v$;
- $N_i = [v_i, v_{i+1}]$: the triangle formed by v_i and v_{i+1};
- $|N_i|$: the area of triangle N_i;
- $V_i = [v_i, v_{i+1}, v_{i+2}]$: the tetrahedron formed by v_i, v_{i+1} and v_{i+2};
- $|V_i|$: the volume of the tetrahedron V_i;
- $\kappa(t)$, $\tau(t)$, $t \in [0, 1]$: the curvature and torsion of curve (1).

Then new conditions for 2×2 and 3×3 diagonal matrices to be Class A matrices are described as follows.

Condition 1. The conditions for a 2×2 diagonal matrix with entries σ_i, $i = 1, 2$ to be a Class A matrix is

$$\sigma_j \geqslant 1, \quad \text{and} \quad 2\sigma_j \geqslant \sigma_k + 1, \quad (j,k) = (1,2), (2,1). \tag{6}$$

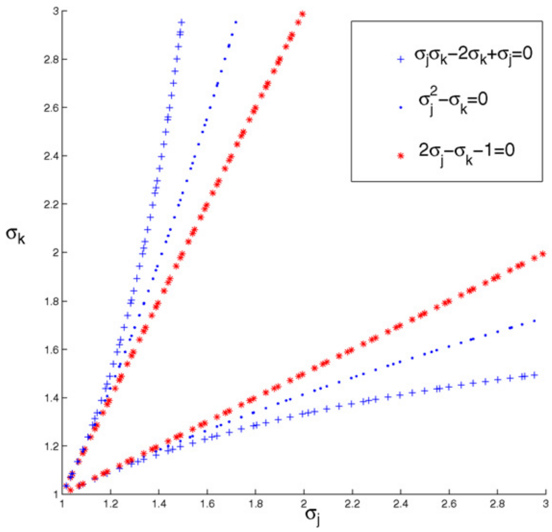

Fig. 1. The boundary curves of the solution sets of inequalities in (9) and (14).

Condition 2. The conditions for a 3×3 diagonal matrix with entries $\sigma_i, i=1,2,3$ to be a Class A matrix is

$$\sigma_j \geqslant 1 \quad \text{and} \quad 3\sigma_j \geqslant \sigma_k+\sigma_s+1, \quad (j,k,s)=(1,2,3),(2,3,1),(3,1,2). \tag{7}$$

We now show that the Class A matrix satisfying the new condition (6) or (10) is subdivision invariant, namely, the curve induced by it holds Property 1.

Theorem 1. *If* 2×2 *diagonal matrix* M *satisfies condition* (6)*, both* T *and* MT^{-1} *will also satisfy condition* (6).

Proof. $\beta_i \geqslant 1$ and $\alpha_i \geqslant 1, i=1,2$ are trivially satisfied. Equivalently, we still need to illustrate that both $2\beta_j-\beta_k-1$ and $2\alpha_j-\alpha_k-1$ are nonnegative under condition (6). We rewrite $2\beta_j-\beta_k-1$ as

$$2\beta_j-\beta_k-1=\frac{[(2\sigma_j-\sigma_k-1)(1-t)+(\sigma_j\sigma_k-2\sigma_j+\sigma_k)t](1-t)}{(1-t+t\sigma_j)(1-t+t\sigma_k)}. \tag{8}$$

It is obvious that the following set leads the expression in (8) nonnegative.

$$\{2\sigma_j-\sigma_k-1, \sigma_j\sigma_k-2\sigma_j+\sigma_k\} \geqslant 0. \tag{9}$$

Furthermore, the set (9) is equivalent to inequality (6) in case of $\sigma_j \geqslant 1$, which can be obtained directly from the images of the solution sets of the two inequalities in (9) (see Fig. 1). Noting that the entries of M and T satisfy

$$2\alpha_j-\alpha_k-1=t(2\sigma_j-\sigma_k-1),$$

which let us complete the proof. □

Theorem 2. *If* 3×3 *diagonal matrix* M *satisfies condition* (10)*, both* T *and* MT^{-1} *will also satisfy condition* (10).

Proof. $\beta_i \geqslant 1$ and $\alpha_i \geqslant 1, i=1,2,3$ are trivially satisfied. Similar to the proof in Theorem 1, $3\beta_j-\beta_k-\beta_s-1$ has the same sign with its numerator

$$(1-t)\left[a_1(1-t)^2+a_2(1-t)t+a_3t^2\right], \tag{10}$$

where

$$a_1=3\sigma_j-\sigma_k-\sigma_s-1,$$
$$a_2=2(\sigma_j-\sigma_k-\sigma_s+\sigma_j\sigma_k+\sigma_j\sigma_s-\sigma_k\sigma_s),$$
$$a_3=\sigma_j\sigma_k\sigma_s+\sigma_j\sigma_k+\sigma_j\sigma_s-3\sigma_k\sigma_s.$$

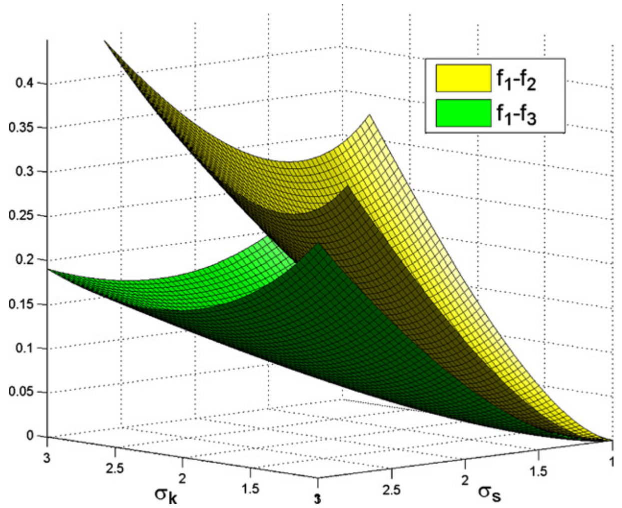

Fig. 2. The image of $f_1 - f_2$ and $f_1 - f_3$.

The expression in "[ ]" of (10) can be considered as a quadratic Bézier function, each coefficient of which exceeding zero makes itself exceed zero. Namely, sufficient conditions for $3\beta_j - \beta_k - \beta_s - 1$ nonnegative are $a_i \geqslant 0, i = 1, 2, 3$, equivalently,

$$\sigma_j \geqslant \max\left\{\frac{\sigma_k + \sigma_s + 1}{3}, \frac{\sigma_k\sigma_s + \sigma_k + \sigma_s}{1 + \sigma_k + \sigma_s}, \frac{3\sigma_k\sigma_s}{\sigma_k\sigma_s + \sigma_k + \sigma_s}\right\}. \tag{11}$$

We denote the three items in "{ }" of (11) as $f_i, i = 1, 2, 3$ orderly. The images of function $f_1 - f_i, i = 1, 2$ have been shown in Fig. 2, which illustrates that f_1 is the largest one among $f_i, i = 1, 2, 3$, i.e., the inequality (11) is equivalent to the inequality (7). Also noting the relation as follows

$$3\alpha_j - \alpha_k - \alpha_s - 1 = t(3\sigma_j - \sigma_k - \sigma_k - 1) \geqslant 0,$$

we complete the proof. □

We still have to verify that the Class A curves under the new condition (6) or (7) will hold Property 2.

Theorem 3. *For a planar Class A Bézier curve, if the matrix M in its derivative* (2) *is a diagonal matrix satisfying condition* (6)*, its curvature will satisfy $\kappa(0) \geqslant \kappa(1)$.*

Proof. Without loss of generality, we suppose that $\kappa(1) \neq 0$. We note that Farin (2006)

$$\kappa(0) = 2\frac{n-1}{n}k_0, \qquad \kappa(1) = 2\frac{n-1}{n}\frac{\|v_{n-1}\|^3}{\|v_n\|^3}k_{n-1}, \tag{12}$$

where $k_i = \frac{|N_i|}{\|v_i\|^3}$. Then, we have

$$\frac{\kappa(0)}{\kappa(1)} = \frac{k_0}{k_{n-1}}\frac{\|v_n\|^3}{\|v_{n-1}\|^3} \geqslant \frac{k_0}{k_{n-1}}\left(\min\{\sigma_1, \sigma_2\}\right)^3 \geqslant \frac{k_0}{k_{n-1}}.$$

Hence,

$$\frac{k_i}{k_{i+1}} = \frac{\|v_{i+1}\|^3}{\sigma_1\sigma_2\|v_i\|^3} = \frac{1}{\sigma_1\sigma_2}\left(\frac{(\sigma_1^{i+1}x_1)^2 + (\sigma_2^{i+1}x_2)^2}{(\sigma_1^i x_1)^2 + (\sigma_2^i x_2)^2}\right)^{\frac{3}{2}} \geqslant 1 \tag{13}$$

will be sufficient for us to complete the proof. The first sign of equality in (13) holds because if one triangle is affinely mapped by a 2×2 matrix, its area will be scaled by the determinant of this matrix, i.e.,$|N_{i+1}| = \sigma_1\sigma_2|N_i|$. By squaring the right part of (13) and subtracting the denominator from the numerator, we obtain a polynomial with respect to x_1^2 and x_2^2. Hence, if all the coefficients are nonnegative, as is

$$\sigma_j^2 \geqslant \sigma_k \geqslant 1, \tag{14}$$

the inequality (13) will hold. The inclusive relation of the solution set of inequality of (14) and (6) can be obtained by referring to Fig. 1, which completes the proof. □

Theorem 4. *For a space Class A Bézier curve, if the matrix M in its derivative* (2) *is diagonal matrix satisfying condition* (7)*, its curvature and torsion will satisfy $\kappa(0) \geqslant \kappa(1)$ and $\tau(0) \geqslant \kappa(1)$, respectively.*

Proof. Without loss of generality, we suppose that $\kappa(1), \tau(1) \neq 0$. Denote $k_i = |N_i|/\|v_i\|^3$ and $q_i = |V_i|/|D_i|^2$ respectively. Similar to the proof in Theorem 3, a sufficient condition for

$$\frac{k_i}{k_{i+1}} = \frac{|D_i|\|v_{i+1}\|^3}{|D_{i+1}|\|v_i\|^3} \geqslant 1,$$

is

$$|D_i|^2\|v_{i+1}\|^6 - |D_{i+1}|^2\|v_i\|^6 \geqslant 0. \tag{15}$$

Actually, the left part of inequality (15) is a polynomial with respect to x_1^2, x_2^2 and x_3^2, therefore, all the coefficients being nonnegative sufficiently make the inequality (15) hold. By omitting the deduction process, we have that coefficients of $x_1^8x_2^2$, $x_1^6x_2^4$, $x_1^6x_2^2x_3^2$ and $x_1^4x_2^4x_3^2$ have the same sign with

$$\sigma_1^4 - \sigma_2^2, \qquad \sigma_1^2 - 1, \quad \sigma_1^6 - \sigma_2^2\sigma_3^2, \qquad \sigma_1^4 - \sigma_3^2,$$

respectively, and others have the similar form because of the symmetry of x_1, x_2 and x_3. Therefore,

$$\sigma_j^3 \geqslant \sigma_k\sigma_s \geqslant \sigma_j \quad \text{and} \quad \sigma_j^2 \geqslant \max\{\sigma_k, \sigma_s\}$$

is a sufficient condition for $\kappa(0) \geqslant \kappa(1)$. For the torsion, we note the fact that any tetrahedron transformed by an 3×3 matrix has its volume being scaled by the determinant of the matrix, hence we have

$$|V_{i+1}| = \sigma_1\sigma_2\sigma_3|V_i|.$$

Since (Farin, 2006)

$$\tau(0) = \frac{3}{2}\frac{n-2}{n}q_0, \qquad \tau(1) = \frac{3}{2}\frac{n-2}{n}\frac{|D_{n-1}|^2}{|D_{n-2}|^2}q_{n-2} \leqslant \frac{3}{2}\frac{n-2}{n}q_{n-2},$$

a sufficient condition for $\tau(0) \geqslant \tau(1)$ is

$$\frac{q_i}{q_{i+1}} = \frac{V_i}{V_{i+1}}\frac{|D_{i+1}|^2}{|D_i|^2} = \frac{|D_{i+1}^2|}{\sigma_1\sigma_2\sigma_3|D_i|^2} \geqslant 1. \tag{16}$$

We deal with (16) in the same way in the proof of Theorem 3, the sufficient condition for which is obtained as

$$\sigma_j\sigma_k \geqslant \sigma_s \geqslant 1.$$

We note that $\sigma_j^2 \geqslant \sigma_k$ and $\sigma_k\sigma_s \geqslant \sigma_j$ can be obtained from

$$\begin{cases} \sigma_j^3 \geqslant \sigma_k\sigma_s, \\ \sigma_j\sigma_s \geqslant \sigma_k, \end{cases} \quad \text{and} \quad \begin{cases} \sigma_k^3 \geqslant \sigma_j\sigma_s, \\ \sigma_s^3 \geqslant \sigma_k\sigma_j, \end{cases}$$

respectively, hence we obtain inequality

$$\sigma_j^3 \geqslant \sigma_k\sigma_s, \tag{17}$$

which is sufficient for us to get $\kappa(0) \geqslant \kappa(1)$ and $\tau(0) \geqslant \tau(1)$. Observing the inclusion relation of the solution sets of the inequalities (7) and (17) in case of $\sigma_i \geqslant 1, i = 1, 2, 3$ in Fig. 3, we complete the proof. □

We modify the Class A conditions, such that the subdivision method can be used to proof the curvature and torsion monotonicity of the Class A curves. However, we only discussed the Class A conditions for symmetric matrices. In practice, non-symmetric matrices satisfying appropriate conditions may also induce curves with monotone curvature and torsion. More works will be addressed to find out the Class A conditions for the non-symmetric matrices.

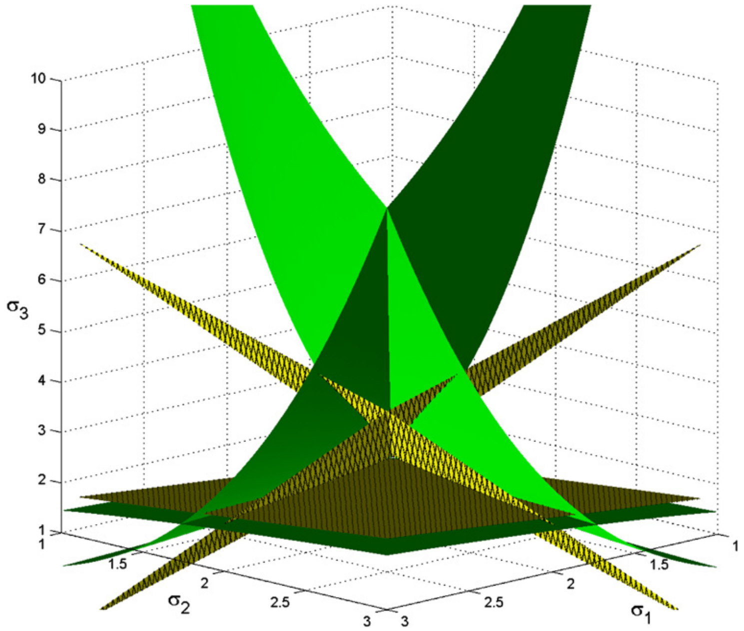

Fig. 3. The inclusive relation of the solution sets of inequalities (7) and (17). The boundary surfaces of the solution sets of inequalities (7) and (17) are colored in green and yellow (with black mesh) respectively. It is obvious that the solution set of inequality (7) is included in the one of inequality (17). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

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